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THE SINGULAR PROBLEM OF THE THEORY OF ELASTICITY FOR A SEMI-INFINITE RECTANGULAR CUTOUT*

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The singular problem of the theory of elasticity is considered in the case of a semi-infinite rectangular cutout on the assumption that the cutout surfaces are free of stresses and that elastic asymptotic behavior of normal rupture cracks obtains at infinity. The solution is constructed by the Kolosov—Muskhelishvili method.

1. Consider the following singular problem of the elasticity theory:

$$\sigma_y = \tau_{xy} = 0, \quad y = \pm 1/2, \quad x \leqslant 0, \quad \sigma_x = \tau_{xy} = 0, \quad |y| \leqslant 1/2, \quad x = 0, \quad \tau_{xy} = 0, \quad v = 0, \quad y = 0, \quad x > 0 \tag{1.1}$$

$$\sigma_y = K_{\rm I} / \sqrt{2\pi x}, \quad y = 0, \quad x \to \infty \tag{1.2}$$

where $\sigma_x, \sigma_y, \tau_{xy}$ are components of the stress tensor, u and v are components of the displacement vector, and $K_{\rm I}$ is the stress itensity coefficient for normal rupture cracks, which defines the stress and strain field at an infinitely distant point.

This boundary value problem belongs to class N in which the Saint Venant principle is not satisfied and a nontrivial solution of homogeneous problems exists (unlike in class S of classic problems of the theory of elasticity in which Saint Venant principle is valid and only a trivial solution of homogeneous problems exists). The general theory of these problems is given in /1/, where it is shown, among other things, the S and N classes are equivalent with respect to strength.

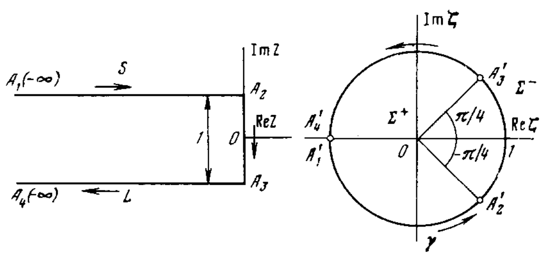

Fig.1

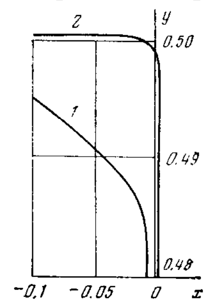

Fig.2

The boundary condition (1.1) may be written thus /2/:

$$\varphi_1(t) + t\overline{\varphi_1'(t)} + \overline{\psi_1(t)} = 0 \text{ on } L \tag{1.3}$$

The contour L is shown in Fig.1.

Functions $\varphi_1(z)$ and $\psi_1(z)$ are holomorphic in S and in accordance with (1.2)

$$\varphi_1(z) = K_{\rm I}\sqrt{z/(2\pi)}, \quad \Psi_1(z) = -z\Phi_1'(z) \quad (z \to \infty), \quad \Phi_1(z) = \varphi_1'(z), \quad \Psi_1(z) = \psi_1'(z), \quad z = x + iy \tag{1.4}$$

We derive the solution of this problem using the method of conformal mapping.

2. Let us determine the function which maps the interior of the unit circle $|\zeta| < 1$ of the plane ζ onto the exterior of the semi-infinite rectangular cutout in the z-plane (Fig.1). Using the Christoffel—Schwartz integral /3/, we obtain

$$z = \omega(\zeta) = \frac{1}{\pi}[\operatorname{Arsh}\xi + \xi\sqrt{\xi^2+1}], \quad \xi = (1+\sqrt{2})(1-\zeta)/(1+\zeta) \tag{2.1}$$

We expand function $z = \omega(\zeta)$ in series in the neighborhood of $\zeta = 0$

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$$\omega(\zeta)=\frac{A}{(1+\zeta)^2}\sum_{k=0}^{\infty}T_k\zeta^k\ ,\quad T_0=B-{}^1/_2,\quad T_1=2B,\quad T_2=B+G_1/2 \tag{2.2}$$

$$T_3=G_1+\frac{G_2}{3},\quad T_k=\frac{G_{k-3}}{k-2}+2\frac{G_{k-2}}{k-1}+\frac{G_{k-1}}{k}\quad(k\geqslant 4)$$

$$B=C/A,\quad C=\pi^{-1}[\mathrm{Arsh}\,(1+\sqrt{2})-2^{3/4}(1+\sqrt{2})^{3/2}]$$

$$A=-\pi^{-1}2^{3/4}(1+\sqrt{2})^{3/2},\quad G_k=P_k+Q_k$$

$$P_k=\sum_{m=0}^{k}A_mA_{k-m}\cos\left(\frac{k-2m}{4}\pi\right)$$

$$A_0=1,\quad A_k=\frac{2k-3}{2k}A_{k-1}\quad(k\geqslant 1)$$

$$Q_1=0,\quad Q_k=\frac{1}{2}\sum_{m=1}^{k-1}(-1)^m(m+1)(m+2)P_{k-m}\quad(k\geqslant 2)$$

Below we use the method of Savin /4/. Rejecting in expansion (2.2) all terms beginning with $T_{n+1}\zeta^{n+1}$, we obtain instead of $\omega(\zeta)$ some function $\omega_n(\zeta)$. The function $\omega_n(\zeta)$ maps the inside of the unit circle $|\zeta|<1$ not onto the specified region S, but onto the close to its region S_n which is the closer to S the greater is n. In conformity with Savin's method we represent the function $\omega(\zeta)$ in the form

$$\omega(\zeta)=\frac{A}{(1+\zeta)^2}\sum_{k=0}^{n}T_k\zeta^k$$

The contours of the cutout corresponding to $n=10$ (curve *1*) and $n=50$ (curve *2*) are shown in Fig.2.

3. We denote regions $|\zeta|<1$, and $|\zeta|>1$, respectively, by Σ^+ and Σ^-, and the circle $|\zeta|=1$ by γ. We take as positive the direction of moving along γ for which region Σ^+ remains to the left.

The boundary condition (1.3) after conformal mapping assumes the form

$$\varphi(\sigma)+\frac{\omega(\sigma)}{\overline{\omega}'(\bar{\sigma})}\overline{\varphi}'(\bar{\sigma})+\overline{\psi}(\bar{\sigma})=0\ \text{on}\ \gamma \tag{3.1}$$

Functions $\varphi(\zeta)=\varphi_1[\omega(\zeta)]$, and $\psi(\zeta)=\psi_1[\omega(\zeta)]$ are homomorphic in Σ^+. As ζ approaches (from inside γ) the point -1, these functions behave in conformity with (1.4) and (2.1) as follows:

$$\varphi(\zeta)=\frac{\sqrt{2}(1+\sqrt{2})}{\pi(1+\zeta)}K_{\rm I},\quad \psi(\zeta)=\frac{1}{2}\varphi(\zeta)\quad(\zeta\to-1) \tag{3.2}$$

We seek functions $\varphi(\zeta)$ and $\psi(\zeta)$ which are holomorphic in Σ^+ of the form

$$\varphi(\zeta)=\frac{1}{1+\zeta}\sum_{k=0}^{\infty}a_k\zeta^k,\qquad \psi(\zeta)=\frac{1}{1+\zeta}\sum_{k=1}^{\infty}c_k\zeta^k \tag{3.3}$$

Multiplying (3.1) by $(\sigma+1)d\sigma/[2\pi i(\sigma-\zeta)]$ and integrating with respect to $\gamma\,(|\zeta|\neq 1)$, we obtain

$$I_1+I_2+I_3=0 \tag{3.4}$$

$$I_1=\frac{1}{2\pi i}\int_\gamma\frac{(\sigma+1)\varphi(\sigma)}{\sigma-\zeta}d\sigma,\quad I_2=\frac{1}{2\pi i}\int_\gamma\frac{(\sigma+1)\omega(\sigma)}{\overline{\omega}'(\bar{\sigma})}\overline{\varphi}'(\bar{\sigma})\frac{d\sigma}{\sigma-\zeta},\quad I_3=\frac{1}{2\pi i}\int_\gamma\frac{(\sigma+1)\overline{\psi}(\bar{\sigma})}{\sigma-\zeta}d\sigma$$

According to Granak's theorem formulas (3.1) and (3.4) are equivalent.

Let us consider the integral I_2. Formula

$$(\sigma+1)\frac{\omega(\sigma)}{\overline{\omega}'(\bar{\sigma})}\overline{\varphi}'(\bar{\sigma})=(\sigma+1)\frac{\omega(\sigma)}{\overline{\omega}'(1/\sigma)}\overline{\varphi}'\left(\frac{1}{\sigma}\right)$$

may be considered as the expression for the boundary value of function

$$(\zeta+1)\frac{\omega(\zeta)}{\overline{\omega}'(1/\zeta)}\overline{\varphi}'\left(\frac{1}{\zeta}\right) \tag{3.5}$$

It is regular in Σ^- and continuous in $\Sigma^-+\gamma$, except at point $\zeta=\infty$, where it has a pole of order $n-1$, and in Σ^- is of the form

$$\frac{(\zeta+1)\,\omega(\zeta)}{\overline{\omega}'(1/\zeta)}\,\overline{\varphi}'\left(\frac{1}{\zeta}\right)=\sum_{k=0}^{n-1}M_k\zeta^k+O\left(\frac{1}{\zeta}\right),\quad M_k=\sum_{r=1}^{n-k}\left[(r-2)\,\bar{a}_{r-1}+r\bar{a}_r\right]b_{r+k} \tag{3.6}$$

$$b_n=T_n,\quad b_m=T_m-\sum_{k=1}^{n-m}\bar{\Gamma}_k b_{m+k},\quad m=(n-1),\ (n-2),\ (n-3),\ldots,0,\ \Gamma_k=(k+1)\eta_k T_{k+1}+(k-2)T_k$$

$$\eta_k=\begin{cases}1, & k\leqslant n-1\\ 0, & k=n\end{cases}$$

where $O(1/\zeta)$ is regular in Σ^- and vanishes at infinity as a part of function (3.5).

Using the properties of the Cauchy integral /2/ and formula (3.6), we obtain

$$I_2=\begin{cases}\sum_{k=0}^{n-1}M_k\zeta^k, & \zeta\in\Sigma^+\\ \sum_{k=0}^{n-1}M_k\zeta^k-\frac{(\zeta+1)\,\omega(\zeta)}{\overline{\omega}'(1/\zeta)}\,\overline{\varphi}'\left(\frac{1}{\zeta}\right), & \zeta\in\Sigma^+\end{cases} \tag{3.7}$$

Let us consider the functions

$$\varphi_2(\zeta)=(\zeta+1)\varphi(\zeta),\quad \overline{\psi}_2(1/\zeta)=(\zeta+1)\overline{\psi}(1/\zeta)$$

The function $(\sigma+1)\varphi(\sigma)$ represents the boundary value of function $\varphi_2(\zeta)$ which is regular in Σ^+ and continuous in $\Sigma^++\gamma$, and function $(\sigma+1)\overline{\psi}(1/\sigma)$ represents the boundary value of function $\overline{\psi}_2(1/\zeta)$ regular in Σ^- and continuous in $\Sigma^-+\gamma$.

From this, using the properties of the Cauchy integral and (3.3), we obtain

$$I_1=\begin{cases}\sum_{k=1}^{\infty}a_k\zeta^k, & (\zeta\in\Sigma^+),\\ 0, & (\zeta\in\Sigma^-),\end{cases}\qquad I_3=\begin{cases}\bar{c}_1, & (\zeta\in\Sigma^+)\\ \bar{c}_1-(\zeta+1)\,\overline{\psi}(1/\zeta), & (\zeta\in\Sigma^-)\end{cases} \tag{3.8}$$

Using formulas (3.4), (3.7) and (3.8) for $\zeta\in\Sigma^+$, we obtain

$$\sum_{k=0}^{\infty}a_k\zeta^k+\sum_{k=0}^{n-1}M_k\zeta^k+\bar{c}_1=0 \tag{3.9}$$

Equating the coefficients at $\zeta^k\ (k=0,1,2,\ldots)$ in both sides of formula (3.9), we obtain a linear homogeneous algebraic system of equations

$$a_0+M_0+\bar{c}_1=0,\quad a_k+M_k=0\quad (k=1,2,\ldots,(n-1)),\quad a_m=0\ (m\geqslant n) \tag{3.10}$$

On the other hand, as $\zeta\to-1$ (from inside γ), we have in conformity with (3.2) and (3.3) one more relation for the unknown constants

$$\sum_{k=0}^{\infty}(-1)^k a_k=\frac{\sqrt{2}\,(1+\sqrt{2})}{\pi}K_{\mathrm{I}} \tag{3.11}$$

Analysis of the system of Eqs. (3.10) and (3.11) shows that all unknown constants are real. We, thus, have a system of $n+1$ equations for the determination of $n+1$ unknown real constants $c_1, a_0, a_1, \ldots, a_{n-1}$. The system of Eqs. (3.10) and (3.11) may be written thus:

$$\sum_{m=1}^{n-1}\{(-1)^m b_{k+1}+\gamma_{m-k}+q_{m-n+k}\left[\gamma_m^*(m-1)\,b_{m+1+k}+mb_{m+k}\right]\}\,a_m=\frac{\sqrt{2}\,(1+\sqrt{2})}{\pi}K_{\mathrm{I}}b_{k+1},\quad k=1,2,\ldots,(n-1)$$

$$a_0=a_1+a_{n-1}/b_n,\quad c_1=-M_0-a_0$$

$$\gamma_m^*=\begin{cases}0, & m=n-k\\ 1, & m<n-k-1\end{cases},\quad \gamma_{m-k}=\begin{cases}1, & m=k\\ 0, & m\neq k\end{cases}$$

$$q_{m-n+k}=\begin{cases}1, & m\leqslant n-k\\ 0, & m>n-k\end{cases}$$

Using formulas (3.3) and (3.10) it is possible to represent the function $\varphi(\zeta)$ as

$$\varphi(\zeta)=\frac{1}{1+\zeta}\sum_{k=0}^{n-1}a_k\zeta^k \tag{3.12}$$

Let us determine the function $\psi(\zeta)$. Formulas (3.4), (3.7), and (3.8) yield

$$c_1 - (\zeta + 1)\,\psi\left(\frac{1}{\zeta}\right) + \sum_{k=0}^{n-1} M_k \zeta^k - \frac{(\zeta+1)\,\omega(\zeta)}{\omega'(1/\zeta)}\,\varphi'\left(\frac{1}{\varepsilon}\right) = 0 \tag{3.13}$$

Finally, using (3.9), (3.12), and (3.13) we obtain the function

$$\psi(\zeta) = -\frac{\omega(1/\zeta)}{\omega'(\zeta)}\,\varphi'(\zeta) - \varphi\left(\frac{1}{\zeta}\right)$$

The stress and displacement field is determined using the Kolosov—Muskhelishvili formulas.

The stresses determined on a computer for $K_I = -1$ and $n = 50$ are tabulated below

$10^3 x$	17	174	244	272	279	307	371	399	406
$10^3 y$	835	619	418	211	0	678	444	220	0
$10^3 \sigma_y$	561	737	760	806	802	577	610	652	654
$10^3 \sigma_x$	245	442	344	134	086	396	378	266	232
$10^3 \tau_{xy}$	226	087	212	115	0	053	112	071	0

The considered problem occurs in investigations of rock burst, in the development of the theory of finite-width cracks, as well as in the investigation of the strength of machine components with rectangular grooves.

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